Exam Linear Algebra II

Thursday 06/04/2023, 15:00–17:00

Exam Hall 1 B22 - H17

1 $(9 = 3 + 1 + 2 + 3$ pts) **Dimension theorem, injective/surjective**

Let P denote the $\mathbb{R}$-vector space consisting of all polynomials over $\mathbb{R}$ in the variable x. The $\mathbb{R}$-linear transformation $T\colon P \to P$ is defined as $T(f) := \int_x^{x+1} f(t)\,dt$.

(a) Show that if $f \in P$ has degree m, then $T(f)$ also has degree m.

(b) Prove that T is injective.

(c) With $P_n \subset P$ the $\mathbb{R}$-subspace consisting of the polynomials of degree $\leq n$, show that as a map from P_n to P_n the transformation T is surjective.

(d) Prove that $T\colon P \to P$ is bijective.

2 $(9 = 3 + 3 + 3$ pts) **Adjoint, orthogonal projection**

Consider the $\mathbb{R}$-inner product space V consisting of all continuous functions $f\colon [0,1] \to \mathbb{R}$; as inner product on V we take $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$. Define the $\mathbb{R}$-linear operator $T\colon V \to V$ by $T(f) = xf$, where xf is the function $[0,1] \to \mathbb{R}$ that maps any $t \in [0,1]$ to $t \cdot f(t) \in \mathbb{R}$.

(a) Prove that the adjoint of T is T itself.

(b) Compute the orthogonal projection of $f \in V$ given by $f(x) = \sqrt{x}$, onto the subspace $P_1 \subset V$ consisting of the real polynomials of degree ≤ 1.

(c) With $P_2 \subset V$ denoting the subspace of all degree ≤ 2 polynomials, define the $\mathbb{R}$-linear transformation $D\colon P_2 \to P_2$ by $D(f) = \frac{df}{dx}$. Prove that the adjoint D^* of D satisfies $D^*(1) = 12x - 6$.

3 (11 = 3 + 3 + 3 + 2 pts) **Positive definite matrices**

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The *k-th leading principal submatrix* A_k of A is the $k \times k$ matrix obtained by removing the last $n-k$ rows and columns of A. It holds that $A_n = A$. The notation $A > 0$ means: A is positive definite.

(a) Assume that $A > 0$. Prove that $A_k > 0$ and $\det(A_k) > 0$ for all $k = 1, 2, \ldots, n$.

(b) Write $A_{i+1} \in \mathbb{R}^{(i+1)\times(i+1)}$ as

$$A_{i+1} = \begin{bmatrix} A_i & b_i \\ b_i^\top & a_i \end{bmatrix},$$

where $b_i \in \mathbb{R}^i$ and $a_i \in \mathbb{R}$. Show that if $A_i > 0$ and $a_i - b_i^\top A_i^{-1} b_i > 0$ then $A_{i+1} > 0$.
Hint: you may want to use the formula:

$$A_{i+1} = \begin{bmatrix} I & 0 \\ b_i^\top A_i^{-1} & 1 \end{bmatrix} \begin{bmatrix} A_i & 0 \\ 0 & a_i - b_i^\top A_i^{-1} b_i \end{bmatrix} \begin{bmatrix} I & A_i^{-1} b_i \\ 0 & 1 \end{bmatrix}.$$

(c) Prove by means of induction that if $\det(A_k) > 0$ for all $k = 1, 2, \ldots, n$ then $A > 0$.

(d) Is the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

positive definite?

4 (7 = 4 + 3 pts) **Singular value decomposition**

Consider the matrix

$$A = \begin{bmatrix} a & -b \\ b & a \\ c & 0 \end{bmatrix}$$

where a, b and c are nonzero real numbers.

(a) Compute a singular value decomposition of A.

(b) Find a best approximation of A of rank at most 1. What is the distance $d(A, M_1)$ from A to the subset of matrices M_1 in $\mathbb{R}^{3\times 2}$ of rank ≤ 1?

4 pts free